Lecture 2: Wealth and Capital Taxation in Ramsey Model: Chamley-Judd

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Application of Diamond-Mirrlees to Intertemporal Taxation

- C. Chamley, "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives", Econometrica, 54 (1986), 607-622. reached by;
- Representative infinitely lived agent with utility $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$
- Simplifying assumptions:
  - single-agent, infinitely lived
  - no uncertainty
- Government:
  - needs to finance g_t
  - chooses
    - taxes to finance g_t
    - government debt b_t

Equilibrium (I)

Consumer problem (1)

$$\max_{c,l\geq 0,k} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t.

$$(1+\tau_{ct})c_t + k_{t+1} + b_{t+1} \leq (1-\tau_{kt})(1+r_t-\delta)k_t + (1-\tau_{lt})w_t l_t + R_t b_t$$

$$k_0 = \bar{k}_0$$

$$l_t \leq 1$$

Equilibrium (II)

- **Firm's problem (2)**

$$\max_{k,l} F(k_t, l_t) - w_t l_t - r_t k_t$$

where F is CRS.

- **Government budget constraint (3)**

$$g_t + R_t b_t \leq \tau_{lt} w_t l_t + \tau_{kt}(1 + r_t - \delta)k_t + \tau_{ct} c_t + b_{t+1}$$

- **Market clearing (4)**

$$c_t + g_t + k_{t+1} \leq F(k_t, l_t) + (1 - \delta)k_1$$

Equilibrium (III)

Definition: CE with taxes $\{\tau_{lt}, \tau_{kt}, \tau_{ct}\}$ and government purchases $\{g_t\}$ is allocations $\{c_t, l_t, k_t, b_t\}$ and prices $\{w_t, r_t\}$ s.t.

1. consumers take prices $\{w_t, r_t\}$ as given and solve consumer problem (1)
2. firms take prices $\{w_t, r_t\}$ as given , solve producer's problem (2) and make 0 profit in equilibrium.
3. government's budget constraint is satisfied (3)
4. markets clear (4)

- Government needs to find $\pi = \{\tau_{lt}, \tau_{kt}, \tau_{ct}\}$ to finance g_t and maximize $\sum_{t=0}^{\infty} \beta^t u(c_t(\pi), l_t(\pi))$ where $c_t(\pi), l_t(\pi)$ are equilibrium allocations with taxes.
- Some observations to what will follow.

Observation 1

- Irrelevance of some taxes: FOCs

$$\frac{u_l(t)}{u_c(t)} = -\frac{(1-\tau_{lt})w_t}{1+\tau_{ct}}$$

$$\frac{\beta u_c(t+1)}{u_c(t)} = \frac{p_{t+1}}{p_t} = \frac{(1+\tau_{ct+1})}{(1+\tau_{ct})(1-\tau_{kt+1})(1+r_{t+1}-\delta)}$$

$$(1-\tau_{kt+1})(1+r_{t+1}-\delta) = R_{t+1}$$

- Too many taxes, can get rid of some.
- We will assume no consumption tax, $\tau_{ct} = 0$ for all t
- Equivalently, we could assume that $\tau_{kt} = 0$ and have

$$\frac{(1+\tilde{\tau}_{ct+1})}{(1+\tilde{\tau}_{ct})} = (1-\tau_{kt+1})$$

- Positive tax on capital and constant tax on consumption is equivalent to zero tax on capital and increasing tax on consumption.

- **Observation 2:** Nothing fancy about dynamics Instead of thinking about period t consumption, think about period 0 consumption of a good with label "t": equivalent to the static commodity taxation problem with infinitely many goods.

- **Observation 3:** Non-distortionary taxation of capital in period 0

  - Note that tax on capital in period 0 does not distort any decisions: equivalent to a lump sum tax.

  - If government could use this tax, it would set it at a very high level to get enought revenues to finance all future g_t

  - Assume (without any justification) that this tax is unavailable to make the problem interesting $\tau_{k0} = 0$

- **Observation 4:** (as before) Many ways to ensure that distortions hold

  - Here: tax gross return on capital $1 + r - \delta$

  - Could instead (as usually done in practice) tax net return $r - \delta$: nothing changes in the analysis.

Primal Approach

- Adding up consumer and government budget constraints gives

$$g_t + c_t + k_{t+1} \leq w_t l_t + (1 + r_t - \delta)k_t = F(k_t, l_t) + (1 - \delta)k_t$$

which is just the resource constraint.

- Now we can just apply the primal approach by finding the implementability constraint:

- **Exercice:** Show that the implementability condition is:

$$\sum_{t=0}^{\infty} \beta^t [u_c(t)c_t + u_l(t)l_t] = u_c(0)R_0 k_0 \quad \text{[ImC]}$$

- **Hint:** Solving b_t forward in the budget constraint of the agent to obtain a Period 0 budget constraint then substitute FOCs into budget constraint.

- so the problem of the planner is

$$\max \sum \beta^t u(c_t, l_t)$$

s.t. the resource constraint and the implementability constraint, ImC.

Zero Capital Taxation in Steady State

- FOCs for $t > 0$

$$\beta^t u_c(t) + \beta^t \mu[u_{cc}(t)c_t + u_c(t) - u_{cl}(t)l_t] = \lambda_t$$

$$\lambda_t = [F_k(t+1) + (1-\delta)]\lambda_{t+1}$$

Therefore

$$\frac{u_c(t) + \mu[u_{cc}(t)c_t + u_c(t) - u_{cl}(t)l_t]}{u_c(t+1) + \mu[u_{cc}(t+1)c_{t+1} + u_c(t+1) - u_{cl}(t+1)l_{t+1}]} = \beta(F_k(t+1) + (1-\delta))$$

- **Theorem: No taxes in steady state**

- **Proof.** Suppose $c_t \to c, k_t \to k$, $l_t \to l$. Then equation above says

$$\beta(F_k + (1-\delta) = 1)$$

Consumer (Euler eq) in steady state

$$(1-\tau_k)\beta(1 + (r-\delta)) = 1$$

and (firm) $r = F_k$. These yield $1 - \tau_k = 1$ so that $\tau_k = 0$. □

A Stronger result

- **Proposition:** Suppose $u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - v(l)$. Then $\tau_{kt} = 0$ for all $t > 1$.
- **Proof:** In this case $u_{cl} = 0$ and CRRA $u_{cc}c = -\sigma u_c$ so that

$$u_c + \mu[u_{cc}c + u_c - u_{cl}l]$$

$$= u_c(1 + \mu[\frac{-\sigma u_c}{u_c} + 1])$$

$$= u_c(1 + \mu[1 - \sigma])$$

  so that the equation above becomes

$$\frac{u_c(t)}{u_c(t+1)} = \beta(F_k(t+1) + (1 - \delta))$$

  this + Consumer euler eq immediately implies that $\tau_{kt} = 0$ for all $t > 1$ □

- Chamley (1986) summary:
  - The optimal tax: tax capital in period $t = 1$, no capital taxes after that
  - Use debt to smooth out tax revenues

Some Difficulties (for now)

- Period 1 capital tax can be huge
  - as time period shrinks (goes to continuous time) capital tax in period 1 becomes the same as capital tax in period 0
  - approaches the level that raises enough revenues to finance all future expenditures, approximates first best
- Plan to tax capital once and never again not time consistent

Judd (1985)

- A model of redistribution:
  - capitalists: own capital stock, do not work, $U(C)$
  - proletariat: supplies labor inelastically, hands-to-mouth, $u(c)$
- Government
  - tax on capital
  - tax on labor $\Leftrightarrow$ lump-sum to workers
  - Pareto weight γ on capitalist
- Two versions:
  - government can trade debt with capitalists to smooth revenues
  - government needs to run budget balance

Judd (1985)

- Feasibility

$$c_t + C_t + k_{t+1} = f(k_t) + (1 - \delta)k_t$$

- Workers budget constraint

$$c_t = w_t + T_t = f(k_t) - f'(k_t)k_t + T_t$$

- Capitalist budget constraint

$$C_t + k_{t+1} + b_{t+1} = R_t(k_t + b_t)$$

Capitalists

- Necessary and sufficient conditions (e.g. Ekeland-Scheinkman, 1986

$$\beta R_t U'(C_t) = U'(C_{t-1})$$

$$\lim_{t\to\infty} \beta^t U'(C_t)[k_{t+1} + b_{t+1}] = 0$$

- Substitute capitalist's budget constraint

$$\beta U'(C_t)(C_t + k_{t+1} + b_{t+1}) = U'(C_{t-1})(k_t + b_t)$$

$$\lim_{t\to\infty} \beta^t U'(C_t)[k_{t+1} + b_{t+1}] = 0$$

Planning Problem

- Planner maximizes weighted sum of utilities

$$\max_{c,C,k,b,T} \sum_{t=0}^{\infty} \beta^t [u(c_t) + \gamma U(C_t)]$$

subject to

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$c_t = f(k_t) - f'(k_t)k_t + T_t$$

$$\beta U'(C_t)(C_t + k_{t+1} + b_{t+1}) = U'(C_{t-1})(k_t + b_t)$$

$$\lim_{t\to\infty} \beta^t U'(C_t)[k_{t+1} + b_{t+1}] = 0$$

- Wants redistribution from capitalist to worker
  - requires sufficiently low γ
  - for simplicity, will sometimes take the extreme: $\gamma \to 0$

Judd Version 1: With Public Debt

- Get rid of lump-sum

$$\max_{c,C,k,b} \sum_{t=0}^{\infty} \beta^t [u(c_t) + \gamma U(C_t)]$$

subject to

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$\beta U'(C_t)(C_t + k_{t+1} + b_{t+1}) = U'(C_{t-1})(k_t + b_t)$$

$$\lim_{t\to\infty} \beta^t U'(C_t)[k_{t+1} + b_{t+1}] = 0$$

- Suppose public debt exists. Substitute away for b_{t+1} and obtain time 0 Implementability condition.

- New planning problem

$$\max_{c,C,k} \sum_{t=0}^{\infty} \beta^t [u(c_t) + \gamma U(C_t)]$$

subject to

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$\sum_{t=0}^{\infty} \beta^t U'(C_t) C_t = U'(C_0) k_0$$

Judd Version 1: With Public Debt

- As before, FOC for C_t:

$$\{\gamma + \mu[\frac{U''(C_t)C_t}{U'(C_t)} + 1]\}U'(C_t) = \lambda_t$$

$$\beta(1 + f'(k_{t+1}) - \delta)\lambda_{t+1} = \lambda_t$$

- So

$$(1 + f'(k_{t+1}) - \delta) = \frac{U'(C_t)}{\beta U'(C_{t+1})} \frac{\gamma + \mu[\frac{U''(C_t)C_t}{U'(C_t)} + 1]}{\gamma + \mu[\frac{U''(C_{t+1})C_{t+1}}{U'(C_{t+1})} + 1]}$$

- In steady state no capital taxes for **any** $\gamma \geq 0$

$$(1 + f'(k_{t+1}) - \delta) = \frac{U'(C^{ss})}{\beta U'(C^{ss})}$$

Judd Version 2: No Public Debt

- Suppose public debt does not exist
- Assume U is CES, $U(C) = C^{1-\sigma}/(1-\sigma)$
- μ_t = multiplier on IC constraint, $\kappa_t = k_t/C_{t-1}$, $\upsilon_t = U'(C_t)/u'(c_t)$

$$\mu_0 = 0$$

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} \upsilon_t} (1 - \gamma \upsilon_t)$$

$$\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + \upsilon_t (\mu_{t+1} - \mu_t)$$

- Judd (1985) studies interior steady state
  - for allocation + multipliers
  - $c_t = c > 0$, $C_t = C > 0$, $k_t = k > 0$, $\mu_t = \mu$
  - Last FOC $\Rightarrow R^* = 1/\beta$
  - Capitalists' Euler $\Rightarrow R = 1/\beta$
  - Hence: **Zero capital tax! .....**

Straub-Werning (2018)

- Something is very fishy here
- Start with Judd V.1, assume CES preferences and $\gamma = 0$
- Immediately

$$\max_{c,C,k} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$\underbrace{(1-\sigma)\sum_{t=0}^{\infty} \beta^t U(C_t)}_{\downarrow \text{ in } C \text{ if } \sigma>1} = \underbrace{U'(C_0)}_{\downarrow \text{ in } C} k_0$$

- There exists sequence $\{C_t\}$ with $C_t \to$ that satisfy the last constraint

Judd Version 1 Revisited

- In the limit $\{C_t\}_t \to 0$ we are solving the first best

$$\max_{c,C,k} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Hence $\{C_t\}_t \to 0$ is optimal, the only feasible way is to set tax $\tau_{kt} = 1$ in some t
- τ_{kt} is a tax on wealth: $\tau_{kt} = 1 \iff$ full expropriation
- it is equivalent to an infinite tax on interest income.
- Why Judd v.1 got wrong result? ...Next lecture
- How about Judd v2 and Chamley (application of Diamond-Mirrlees)?

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