Lecture 3: Wealth and Capital Taxation Revisited: Straub-Werning

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Straub-Werning (2018)

- Something is very fishy here
- Start with Judd V.1, assume CES preferences and $\gamma = 0$
- Immediately

$$\max_{c,C,k} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$\underbrace{(1-\sigma)\sum_{t=0}^{\infty} \beta^t U(C_t)}_{\downarrow \text{ in } C \text{ if } \sigma>1} = \underbrace{U'(C_0)}_{\downarrow \text{ in } C} k_0$$

- There exists sequence $\{C_t\}$ with $C_t \to 0$ that satisfy the last constraint

Judd Version 1 Revisited

- In the limit $\{C_t\}_t \to 0$ we are solving the first best

$$\max_{c,C,k} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Hence $\{C_t\}_t \to 0$ is optimal, the only feasible way is to set tax $\tau_{kt} = 1$ in some t
- τ_{kt} is a tax on wealth: $\tau_{kt} = 1 \iff$ full expropriation
- it is equivalent to an infinite tax on interest income.
- How about Judd v2 and Chamley (application of Diamond-Mirrlees)?

Judd Version 2

- Assume $U(C) = C^{1-\sigma}/(1-\sigma)$
- μ_t = multiplier on IC constraint, $\kappa_t = k_t/C_{t-1}$, $\upsilon_t = U'(C_t)/u'(c_t)$

$$\mu_0 = 0$$

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} \upsilon_t} (1 - \gamma \upsilon_t)$$

$$\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + \upsilon_t (\mu_{t+1} - \mu_t)$$

- Judd (1985) studies interior steady state
  - for allocation + multipliers
  - $c_t = c > 0,\ C_t = C > 0,\ k_t = k > 0,\ \mu_t = \mu$
  - Last FOC $\Rightarrow R^* = 1/\beta$
  - Capitalists' Euler $\Rightarrow R = 1/\beta$
  - Hence: **Zero capital tax! ......**

Judd Version 2 Revisited

- **... or not ???**
- non convergence of allocation (cycles) ?
- convergence to non-interior steady state?
- non convergence of multipliers?

Judd Version 2 Revisited

- **... or not ???**
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Log case

- Simple special case: $\sigma = 1$, $U(C) = \log C$
  $\Rightarrow$ constant savings rate β,

$$C_t = (1-\beta)R_t k_t$$

$$k_{t+1} = \beta R_t k_t = \frac{\beta}{1-\beta} C_t$$

- Substitute out C_t in planning problem (with $\gamma = 0$)

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + \frac{1}{\beta} k_{t+1} + g \leq f(k_t) + (1-\delta)k_t$$

- **Like a neoclassical growth model, with higher cost of capital!**

Log case

- Converges to unique interior steady state
- Planner's Euler: $R^* = 1/\beta^2$
- Capitalists: $R = 1/\beta$

$$\text{tax} = 1 - \frac{R}{R^*} = 1 - \beta$$

- Why positive tax?
  - multipliers do not converge (Reinhorn 2002)
- Is this specific to log preferences?
  - Lansing (1999): Yes, "knife-edged"
- Werning-Straub: **No!** Positive capital taxation for all $\sigma \geq 1$!

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First order conditions

- Take planning problem FOCs with $\gamma = 0$
- Suppose allocation did converge to interior steady state (c, C, k)
- Law of motion for multiplier μ_t of capitalists' IC

$$\mu_0 = 0$$
$$\mu_{t+1} = \mu_t \underbrace{\left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1\right)}_{\to \text{const} > 1} + \underbrace{\frac{1}{\beta \sigma \kappa_{t+1} \upsilon_t}}_{\to \text{const} > 0}$$

Hence μ_t diverges to $+\infty$, and so does $\mu_{t+1} - \mu_t$

- FOC for capital gives a **contradiction:**

$$\underbrace{\frac{u'(c_{t+1})}{u'(c_t)}}_{\to 1} \underbrace{(f'(k_{t+1}) + 1 - \delta)}_{\to \text{const}} = \frac{1}{\beta} + \underbrace{\upsilon_t(\mu_{t+1} - \mu_t)}_{\to +\infty}$$

Positive long run capital taxation

- This proves:
- **Proposition 2: S-W** If $\sigma > 1$, the optimal allocation **cannot** be converging to the zero capital tax steady state
  - ... or in fact, *any* other interior steady state
- Next result shows the system converges to a *non*-interior steady state!
- **Proposition 3: S-W** If $\sigma > 1$, the optimal allocation satisfies

$$c_t \to 0 \qquad k_t \to k_g \qquad C_t \to \frac{1-\beta}{\beta} k_g$$

$$\text{tax} = 1 - \frac{R_t}{R_t^*} \to \mathcal{T}_g > 0$$

where $\mathcal{T}_g \to 100\%$ as $g \to 0$.
  - Here k_g is the lowest feasible steady state capital stock, $\frac{1}{\beta} k_g + g = f(k_g) + (1-\delta) k_g$

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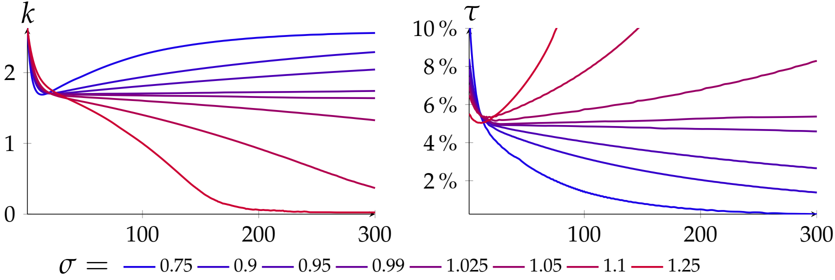
Intuition

- Intuition: Incentivizing capitalists' savings behavior through anticipatory effects
  - Start with a constant tax
  - Announce a tax increase in the far future
  - IES $< 1 \Rightarrow$ capitalists increase savings today
  - ... which is great if capital is taxed today!
- Rationalizes why the planner likes an *positive* slope for capital taxes!
- Reverse holds for IES > 1: tax *decreases* to zero

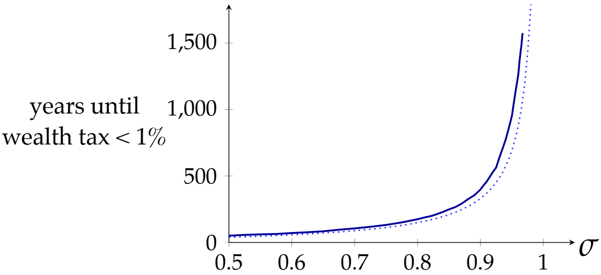
Capital and taxes for various IES's

- One can solve a recursive version of the planning problem
- Here: take $\gamma = 0$ and let σ range from 0.75 to 1.25
- Left graph: capital stock k_t, right graph: wealth tax $\mathcal{T}_t$



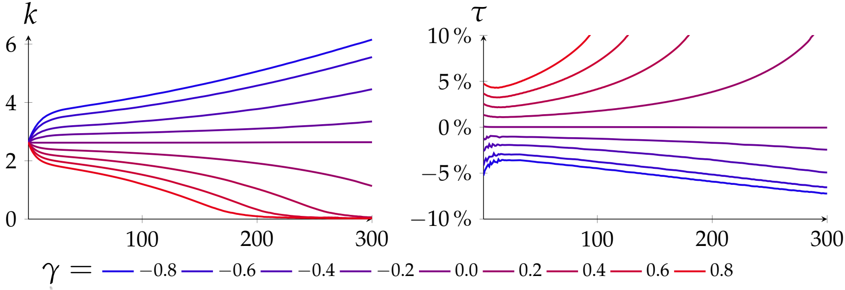
Slow convergence for $\sigma < 1$ (IES > 1)

- For $\sigma < 1$, tax does converge zero, but **convergence is slow**:



Capital and taxes for various degrees of redistribution

- Solve same planning problem, but now keep σ fixed at 1.25 and vary γ
  - normalize γ such that zero = no redistribution at zero tax steady state
  - $\gamma < 0 \rightarrow$ redistribution towards capitalists
  - $\gamma > 0 \rightarrow$ redistribution towards workers
- Illustrates that Prop 2 and 3 are robust to nonzero γ
- A more formal result is in Werning-Straub



General savings functions

- Proposition 2 can be generalized to almost arbitrary savings functions for capitalists
- Again assume $\gamma = 0$ for simplicity
- Suppose having time t income I_t, capitalists save exactly $S(I_t, R_{t+1}, R_{t+2}, \ldots)$
  - naturally depends on future interest rates $\{R_{t+1}, R_{t+2}, \ldots\}$
  - Assume S increases in I_t and weakly decreases in future interest rates
- **Proposition 4: S-W** Optimal tax rates **cannot** converge to zero (or anything negative)

Binding bounds

- in Chamley (1986) we found that period 1 tax can be large. So set upper bound on taxes
- If bounds on capital tax rates are asymptotically slack, then long run capital tax is zero.
- Was it reasonable to assume capital tax bounds do not bind indefinitely in Chamley (1986)?
- Move to continuous time (for simple bang-bang optimal tax policies)
- Assume separable isoelastic utility

$$\int_0^\infty e^{-\rho t} u(c_t, n_t) dt \qquad u(c, n) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{n^{1+\zeta}}{1+\zeta}$$

- Resource constraint

$$c_t + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t$$

- Budget constraint and implementability as before

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Planning problem

- Planner solves

$$\max \int_0^\infty e^{-\rho t} u(c_t, n_t) dt$$

subject to RC and IC

$$c_t + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t$$

$$\int_0^\infty e^{-\rho t} \left(u_{ct} c_t + u_{nt} n_t \right) = u_{c0}(k_0 + b_0)$$

and bounds on taxes (assume $\overline{\tau} = 1$ for simplicity)

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(r_t - \rho)$$

$$r_t = (1 - \tau_t) \left(f_k(k_t, n_t) - \delta \right)$$

$$\tau_t \leq \overline{\tau}$$

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$$\frac{\dot{c}_t}{c_t} \geq -\frac{\rho}{\sigma}$$

Chamley (1986, Theorem 2)

- **Chamley (1986, Theorem 2):** Suppose $\overline{\tau} = 1$. Then there exists a time $T < \infty$ such that
  - $\tau_t = \overline{\tau}$ for $t < T$
  - $\tau_t = 0$ for $t > T$
- Bang-bang due to continuous time
- But why can T not be infinite?
  - Chamley's (1986) proof: "The bounds cannot be binding forever or marginal utility would grow to infinity, which is absurd"
- Next: Nothing absurd here ...

Positive long run capital taxation

- **Proposition 7: S-W** Take $\overline{\tau} = 1$ and $\sigma > 1$. Fix initial capital k_0. Then there exist $\underline{b} < \overline{b}$ such that
  - if $b_0 \in [\underline{b}, \overline{b}] \Rightarrow T = \infty$!
- **For sufficiently high levels of initial debt b_0, the bounds on capital taxes bind forever!**
- Can construct specific analytically tractable examples (see Straub-Werning)

Proof idea

- Planning problem with current value multipliers

$$\max \int_0^\infty e^{-\rho t} u(c_t, n_t) dt$$

$$c_t + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t \qquad (\lambda_t)$$

$$\int_0^\infty e^{-\rho t} (u_{ct} c_t + u_{nt} n_t) \geq u_{c0}(k_0 + b_0) \qquad (\mu)$$

$$\dot{c}_t \geq -\frac{\rho}{\sigma} c_t \qquad (\eta_t)$$

- Note: $b_0 \uparrow \Rightarrow$ gov. needs to tax more $\Rightarrow$ IC constraint tighter $\Rightarrow \mu \uparrow$
- In fact: As b_0 approaches highest feasible debt level $\overline{b}$, $\mu \nearrow +\infty$
- Now pick $\sigma > 1$ and suff. high b_0 (hence high μ), and prove $T = \infty$

Proof idea (2)

- Consider FOC for consumption

$$\dot{\eta}_t - \rho\eta_t = \eta_t \frac{\rho}{\sigma} + \lambda_t - (1 - \mu(\sigma - 1))\, u_{ct}$$

where tax bound $\tau_t = \overline{\tau}$ binds if $\eta_t < 0$

- Note that if $T < \infty \Rightarrow \eta_t = \dot{\eta}_t = 0 \; \forall t > T$, implying for such t

$$\underbrace{\lambda_t}_{\geq 0} = \underbrace{(1 - \mu(\sigma - 1))}_{\text{possibly } < 0\,!} \underbrace{u_{ct}}_{>0}$$

- **This is impossible if $\sigma > 1$ and μ sufficiently large!**
- Hence indefinite capital taxation, $T = \infty$, is optimal in those cases

Judd (1999)

- Representative agent model as in Chamley (1986)
- Does not assume convergence of allocation
  - Instead assumes multiplier Λ_t is bounded
- **Result:** long-run average tax on capital is zero
- **Intuition:** exploding consumption taxes are infinitely distortionary
- **But, are bounds on endogenous multiplier reasonable?**

Model

- Use same continuous time planning problem as for Chamley

$$\max \int_0^\infty e^{-\rho t} u(c_t, n_t) dt$$

$$c_t + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t$$

$$\int_0^\infty e^{-\rho t} \left(u_{ct} c_t + u_{nt} n_t \right) = u_{c0}(k_0 + b_0)$$

$$\frac{\dot{c}_t}{c_t} \geq -\frac{\rho}{\sigma}$$

- Call $e^{-\rho t} u_{ct} \Lambda_t$ the multiplier on the resource constraint

Judd (1999)

- Planner's first order condition
$$\frac{\dot{\Lambda}_t}{\Lambda_t} = r_t - r_t^*$$
- If Λ_t converges: zero tax!
- **Judd (1999):** If there are $0 < \underline{\Lambda} < \overline{\Lambda}$ with $\Lambda_t \in [\underline{\Lambda}, \overline{\Lambda}]$, then average capital tax goes to zero,
$$\frac{1}{t}\int_0^t (r_s - r_s^*)ds \to 0$$
- Follows immediately from imposing the bounds on Λ_t !
- Are the bounds reasonable?
No, see the above positive tax result for Chamley: There, $\Lambda_t \to 0$

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Judd (1999): Alternative interpretation

- First order condition for capital implies

$$\mathrm{MRS}^{\text{planner}}_{t,t+s} = \mathrm{MRT}_{t,t+s} = \exp\left\{-\int_0^s r^*_{t+\tilde{s}} d\tilde{s}\right\}$$

- Using the agent's Euler condition

$$\mathrm{MRS}^{\text{planner}}_{t,t+s} = \mathrm{MRS}^{\text{agent}}_{t,t+s} \frac{\Lambda_{t+s}}{\Lambda_t}$$

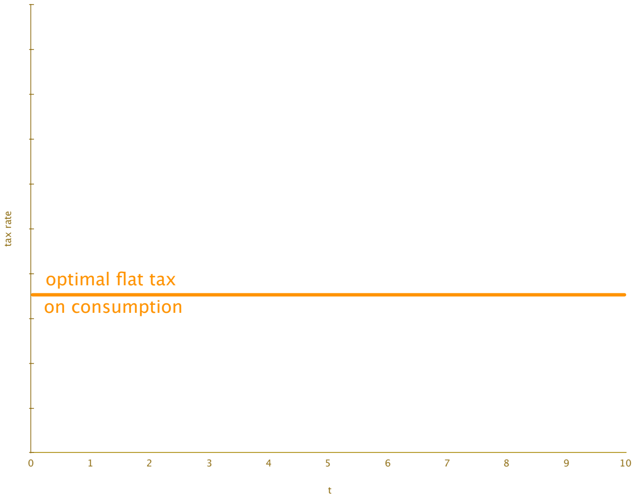
- Hence, **assuming Λ_t converges (or is bounded) is essentially assuming the result!**

Consumption tax intuition

- Common intuition for zero capital tax results:
  - an ever-rising tax on consumption is infinitely distortionary
  - hence not optimal
- **But:** here, there are **bounds on capital taxation**
  - this is **not** a standard Diamond-Mirrlees economy
  - the bounds force equivalent consumption taxes to be low initially...
  - ... which may make ever-rising consumption taxes a third-best

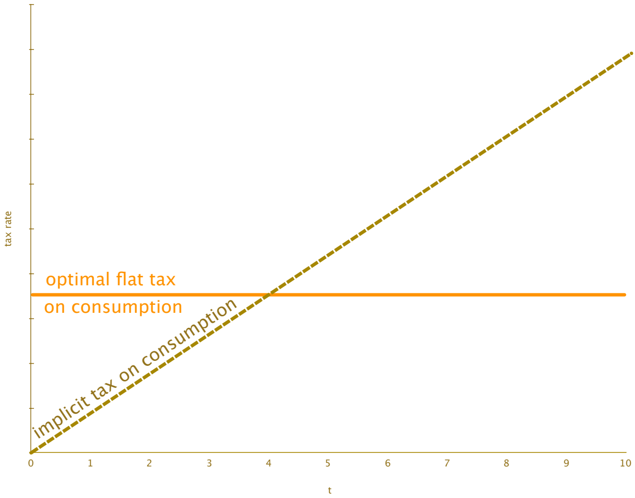
Graphical illustration

- **Flat** optimal consumption tax path without any capital tax bound



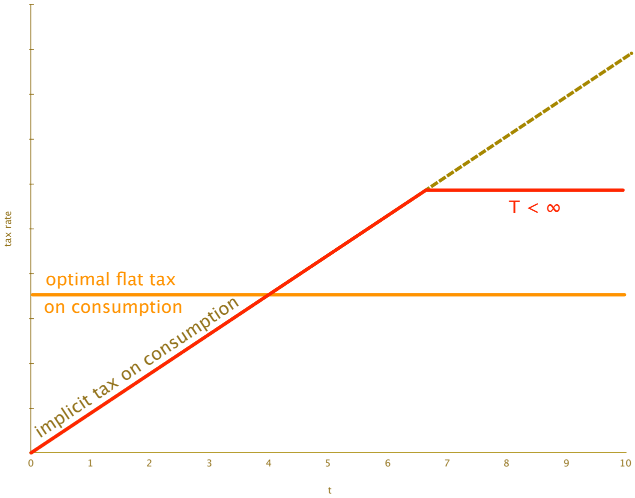
Graphical illustration

- Capital tax bound is equivalent to restriction on consumption taxes



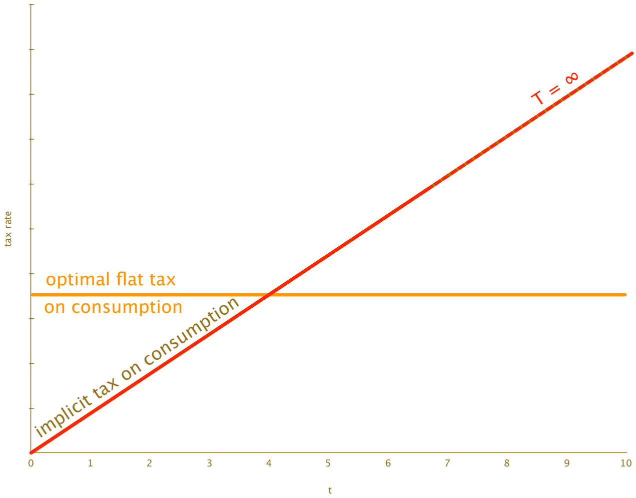
Graphical illustration

- For example, one could pick a consumption tax path like this...



Graphical illustration

- ... but it might well turn out that $T = \infty$ is actually optimal here



Consumption tax intuition more formally

- Straub-Werning adopt a simple linear technology framework to show that
  - yes, longer capital taxation creates larger distortions...
  - ...but indefinite capital taxation is **not** "infinitely distortionary"
- Hence there is no reason for why indefinite capital taxation cannot be optimal, *despite* an ever-increasing consumption tax!

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